Computational Validation of a 2-D Semi-empirical Model for Inductive Coupling in a Conical Pulsed Inductive Thruster

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Outline of Talk

- Pulsed Inductive Plasma Thrusters

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  - Finite element results indicate limit to applicability
- Non-dimensional Analysis
- Conclusions

Pulsed Inductive Plasma Thrusters

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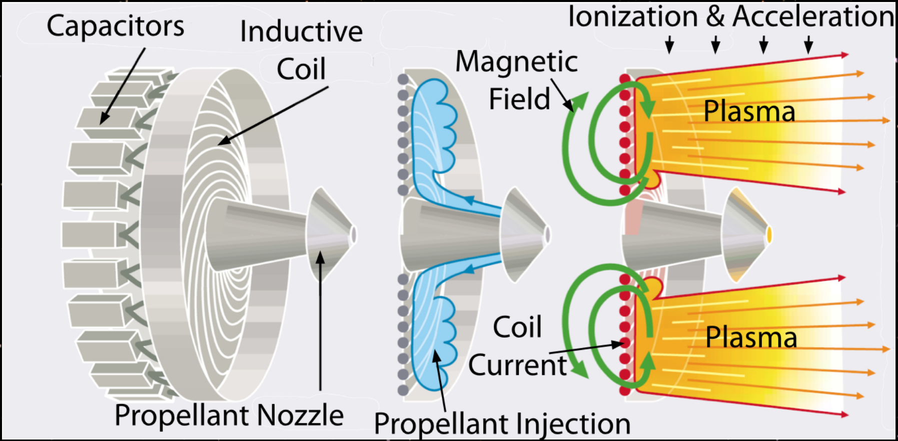
Lack of Cavity Decreases Propellant Utilization

Idealized thruster operation :



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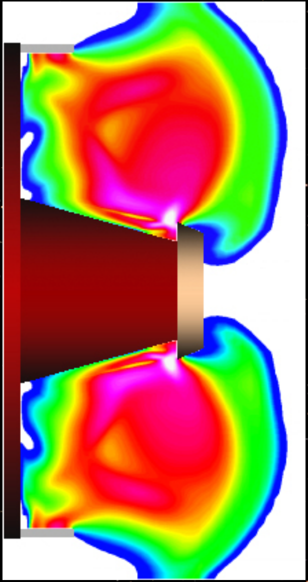
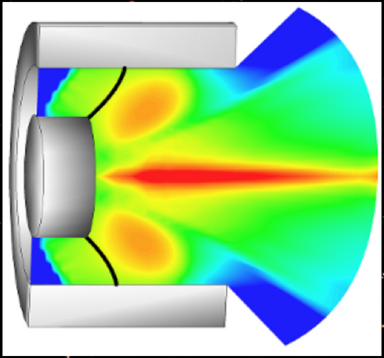


vs.



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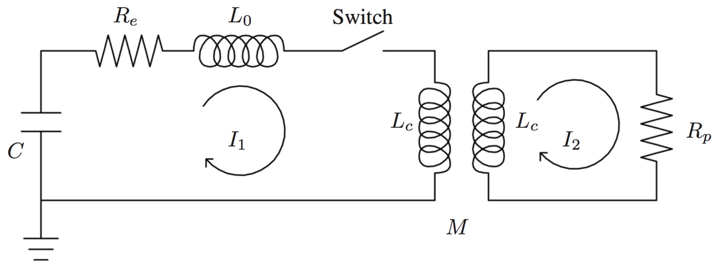
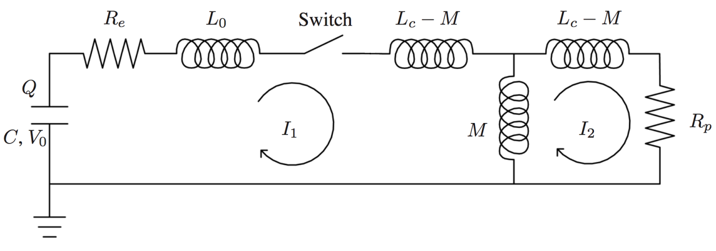
Analytical Model

Model Thruster-Plasma System as Circuits





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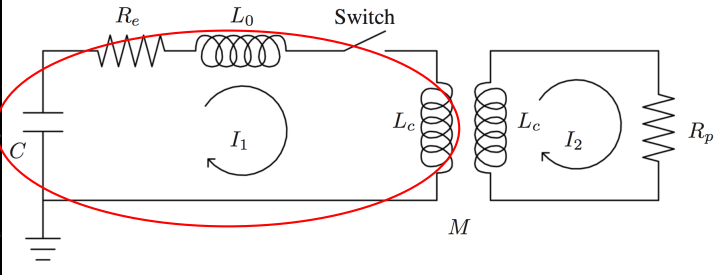

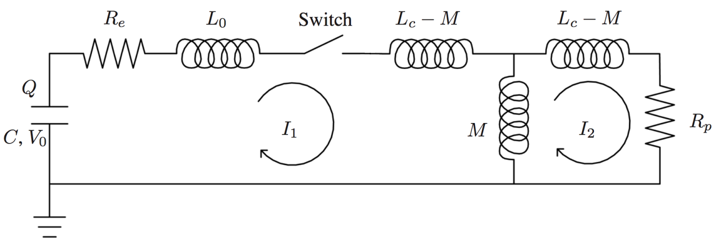

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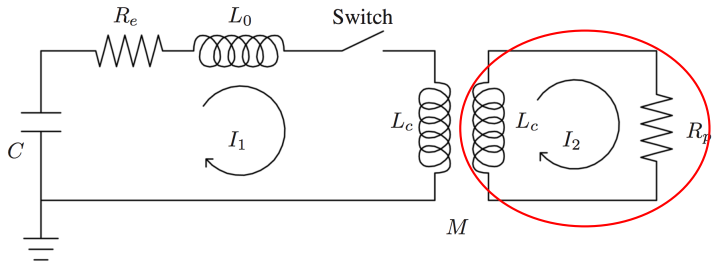

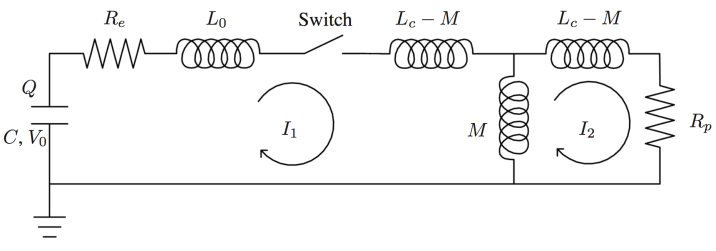

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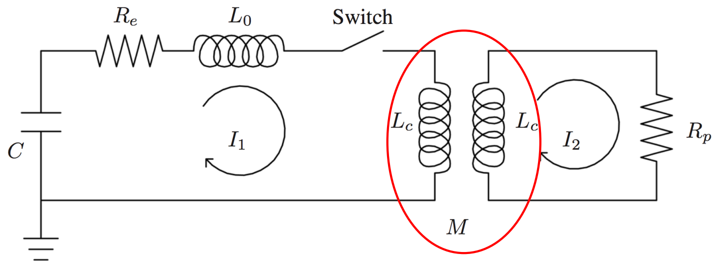

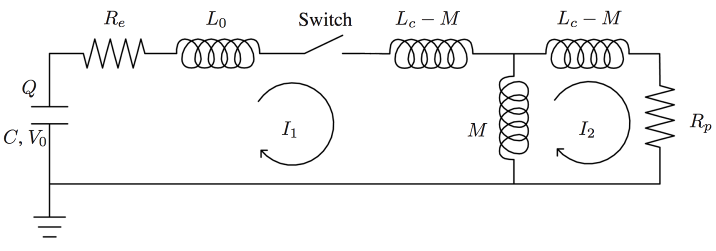

Governing Equations via Kirchhoff's Law

$$\frac{dI_1}{dt} = \frac{L_C V - L_C R_e I_1 - M R_p I_2 + \left(L_C I_2 + M I_1\right) \frac{dM}{dt}}{L_C \left(L_0 + L_C\right) - M^2}$$

$$\frac{dI_2}{dt} = \frac{M \frac{dI_1}{dt} + I_1 \frac{dM}{dt} - R_p I_2}{L_C}$$

$$\frac{dV}{dt} = -\frac{I_1}{C}$$

Equations Governing Current Sheet Motion

$$L_{tot} = L_0 + L_C - \frac{M^2}{L_C}$$

$$L_{tot}(\bar{r}, z) = L_0 + L_C \left(1 - \exp\left(-z/z_0\right) \left(\frac{\bar{r}}{r_{coil}}\right)^N\right)$$

$$M = L_C \exp\left(-\frac{z}{2z_0}\right) \left(\frac{\bar{r}}{r_{coil}}\right)^{N/2}$$

$$F_i = \frac{I^2}{2} \frac{\partial L}{\partial x_i}$$

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Circuit Eqns. $\boxed{L_{tot} = L_0 + L_C - \frac{M^2}{L_C}}$

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Empirical

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$$\frac{dv_z}{dt} = \frac{\left[\frac{L_C I_1^2}{2z_0}\exp\left(-\frac{z}{z_0}\right)\left(\frac{\bar{r}}{r_{coil}}\right)^N\right]}{m_{bit}}$$

$$\frac{dv_r}{dt} = \frac{\left[P_2 2\pi\bar{r} l_{coil} - \frac{L_C I_1^2 N}{2\overline{r_{coil}}^N}\exp\left(-\frac{z}{z_0}\right)(\bar{r})^{N-1}\right]}{m_{bit}}$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left[\mathcal{M}^2 - 1\right]$$

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Model Relies on Semi-Empirical Expression

$$L_{tot}(\bar{r}, z) = L_0 + L_C \left(1 - \exp\left(-z/z_0\right) \left(\frac{\bar{r}}{r_{coil}} \right)^N \right)$$

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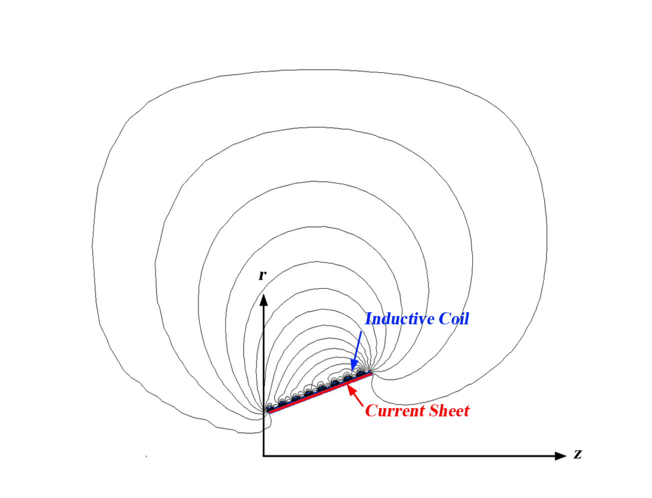
Applicable to all inductive coil geometries?

Computational Validation

Simulation Configuration for Radial Compression



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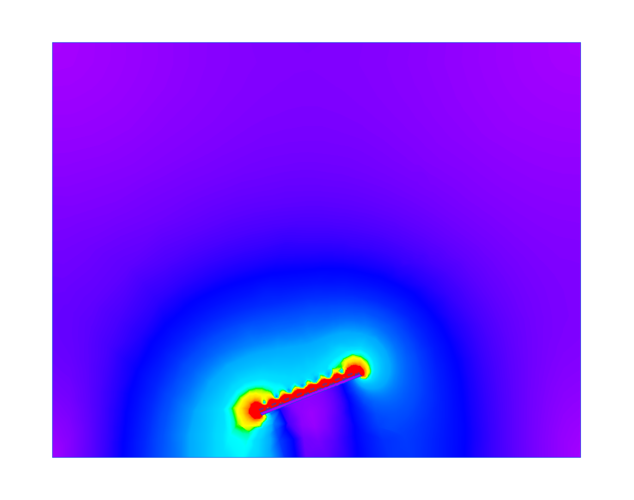
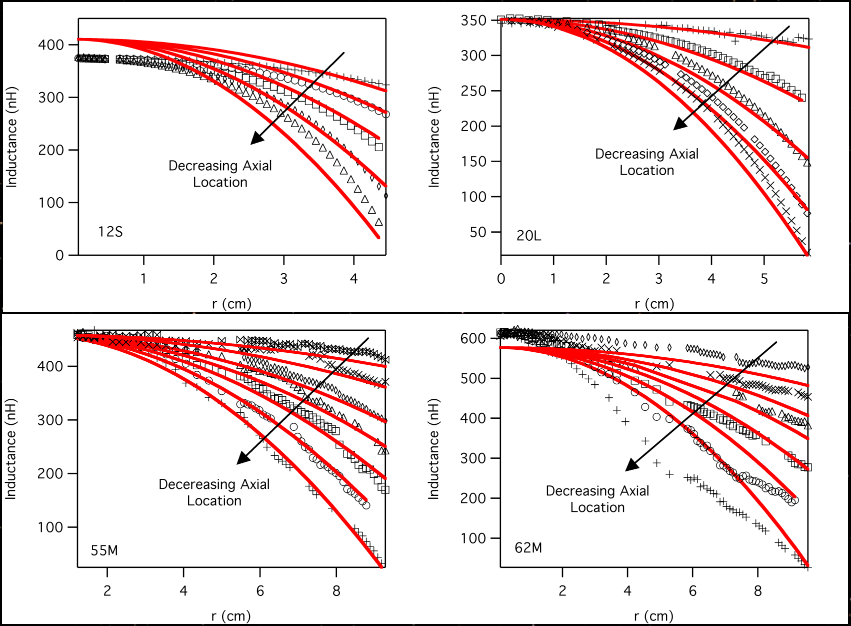
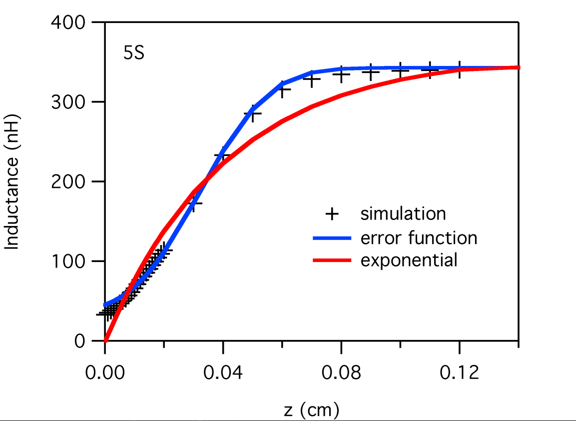


Good Agreement from 20°-55°



Error Function Better Fit at Angles less than 20°



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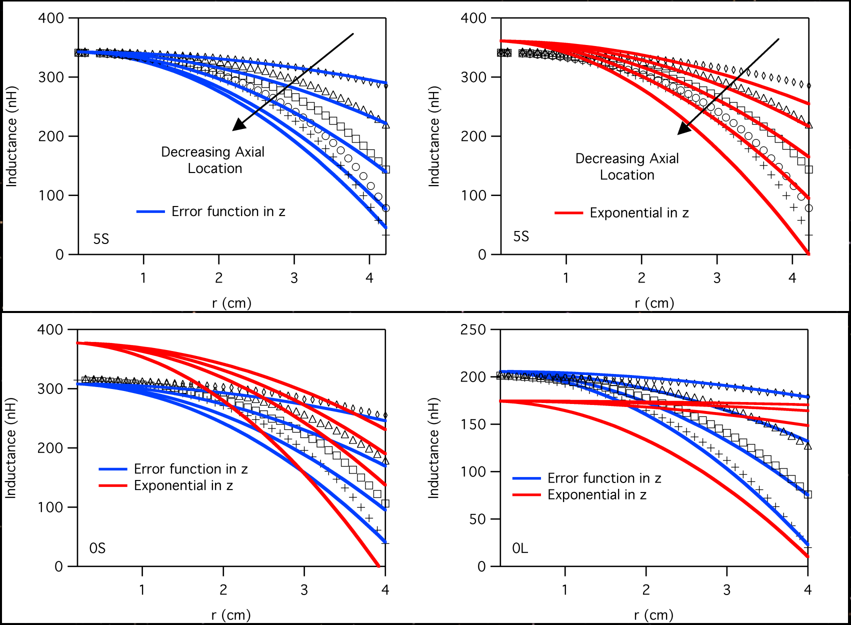

Non-dimensional Analysis

Substitutions

$$I_1^* = \frac{1}{V_0}\sqrt{\frac{L_C}{C}} I_1 \qquad I_2^* = \frac{1}{V_0}\sqrt{\frac{L_C}{C}} I_2$$

$$V^* = \frac{V}{V_0} \qquad M^* = \frac{M}{L_C}$$

$$v_z^* = \frac{\sqrt{L_0 C}}{z_0} v_z \qquad z^* = \frac{z}{z_0}$$

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Resulting Non-dimensional Equation Set

$$\frac{dI_1^*}{dt^*} = \left[L^* V^* + \left(M^* I_1^* + I_2^*\right) dM^* / dt^*\right] / \left(L^* + 1 - {M^*}^2\right) - \left[\psi_1 L^* I_1^* - \psi_2 L^* I_2^* M^*\right] / \left(L^* + 1 - {M^*}^2\right)$$

$$\frac{dI_2^*}{dt^*} = M^* \frac{dI_1^*}{dt^*} + I_1^* \frac{dM^*}{dt^*} - I_2^* L^* \psi_2$$

$$\frac{dV^*}{dt^*} = -I_1^*$$

$$\frac{dr^*}{dt^*} = v_r^*$$

$$\frac{dz^*}{dt^*} = v_z^*$$

Resulting Non-dimensional Equation Set

$$\boxed{\begin{aligned}\frac{dI_1^*}{dt^*} &= \left[L^* V^* + \left(M^* I_1^* + I_2^*\right) dM^*/dt^*\right] / \left(L^* + 1 - {M^*}^2\right) \\ &\quad - \left[\psi_1 L^* I_1^* - \psi_2 L^* I_2^* M^*\right] / \left(L^* + 1 - {M^*}^2\right)\end{aligned}}$$

$$\frac{dI_2^*}{dt^*} = M^* \frac{dI_1^*}{dt^*} + I_1^* \frac{dM^*}{dt^*} - I_2^* L^* \psi_2$$

$$\frac{dV^*}{dt^*} = -I_1^*$$

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Resulting Non-dimensional Equation Set

$$\frac{dM^*}{dt^*} = \frac{N}{2} {r^*}^{\frac{N}{2}-1} v_r^* \exp(-\frac{z^*}{2}) - \frac{1}{2} {r^*}^{\frac{N}{2}} v_z^* \exp(-\frac{z^*}{2})$$

$$\frac{dv_r^*}{dt^*} = \lambda P^* r^* - \phi {l_1^*}^2 {r^*}^{N-1} \exp(-z^*)$$

$$\frac{dv_z^*}{dt^*} = \alpha {l_1^*}^2 {r^*}^N \exp(-z^*)$$

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Non-dimensional Parameters

$$\alpha = \frac{V_0^2 C^2 L_C}{2 m_{bit} z_0^2}$$

$$\psi_1 = R_e \sqrt{\frac{C}{L_0}}$$

$$\phi = \frac{V_0^2 C^2 L_C}{2 m_{bit} \overline{r_{coil}}^2}$$

$$\psi_2 = R_p \sqrt{\frac{C}{L_0}}$$

$$\lambda = \frac{L_0 C P_1 2 \pi l_{coil}}{2 m_{bit}}$$

$$\Xi = \frac{4\gamma}{\gamma + 1} \frac{m_i}{\gamma k T_1} \frac{1}{r_{coil}^2 L_0 C}$$

$$L^* = \frac{L_0}{L_C}$$

New Non-dimensional Parameters

$$\alpha = \frac{V_0^2 C^2 L_C}{2 m_{bit} z_0^2}$$

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Physical Meaning of Scaling Parameters

$$\alpha = \frac{C^2 V_0^2 L_C}{2 m_{bit} z_0^2} = \frac{1}{8\pi^2} \frac{C V_0^2/2}{m_{bit} v_z^2/2} L^* \left(\frac{2\pi\sqrt{L_0 C}}{L_0/\dot{L}_z} \right)^2$$

$$\phi = \frac{C^2 V_0^2 L_C}{2 m_{bit} r_{coil}^2} = \frac{1}{8\pi^2} \frac{C V_0^2/2}{m_{bit} v_r^2/2} L^* \left(\frac{2\pi\sqrt{L_0 C}}{L_0/\dot{L}_r} \right)^2$$

Physical Meaning of Scaling Parameters

Circuit Timescale

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Axial Decoupling Timescale

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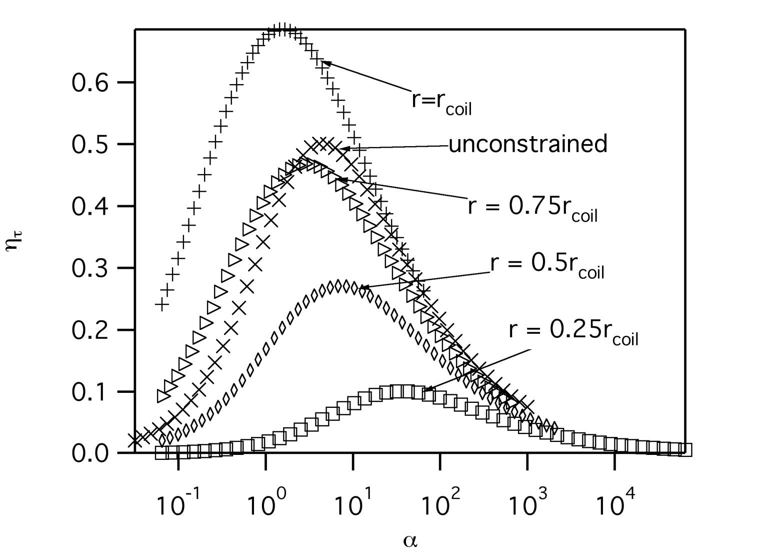
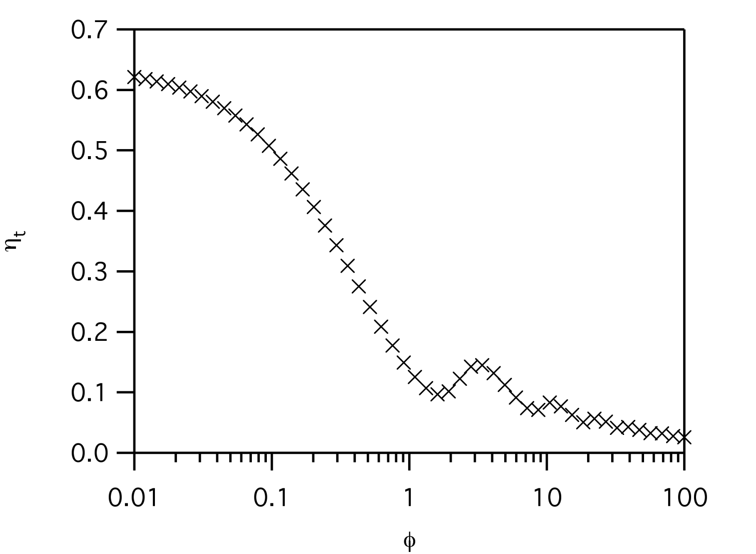
Radial Decoupling Timescale

Radial Motion Shifts Peak in Thrust Efficiency

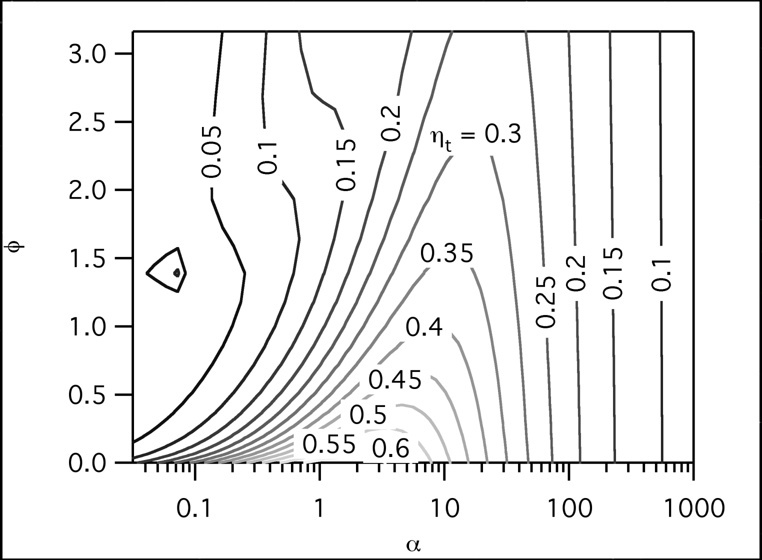


Thrust Efficiency Maximum at Lower Values of Phi



Approved for public release. Distribution is unlimited.

Combined Effects of α and ϕ on η_t



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- This leads to dynamic impedance matching at longer characteristic circuit times
- Thrust efficiency is maximized when the axial decoupling timescale is shorter than the radial decoupling timescale

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Questions?